Vector bundles, connections, and parallel transport

Board notes, Talk 1 (Seminar *Mathematical Gauge Theory*). Based on Differential Geometry - Connections, Curvature, and Characteristic Classes" by Loring Tu

Part I: Vector bundles

Motivation

Fields. $p \mapsto s(p) \in E_p$, a vector space depending on p $\quad\rightsquigarrow\quad$ vector bundle, section.

Problem. $s(q) - s(p)$ is not defined, since $E_q \neq E_p$ $\quad\rightsquigarrow\quad$ connection ∇, parallel transport.

Definition

Definition 7.1. A C^∞ surjection $\pi\colon E \to M$ is a C^∞ *vector bundle of rank* r if

(i) for every $p \in M$, the set $E_p := \pi^{-1}(p)$ is a real vector space of dimension r;

(ii) every point $p \in M$ has an open neighbourhood U such that there is a fibre-preserving diffeomorphism
$$\varphi_U\colon \pi^{-1}(U) \to U \times \mathbb{R}^r$$
that restricts to a linear isomorphism $E_p \to \{p\} \times \mathbb{R}^r$ on each fibre.

E is the *total space*, M the *base space*, E_p the *fibre above* p. U as in (ii) is a *trivialising open set* and φ_U a *trivialisation* of $\pi^{-1}(U)$. A *trivialising open cover* is an open cover $\{U_\alpha\}$ of M together with trivialisations $\varphi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^r$.

Fibre-preserving means $\mathrm{pr}_1 \circ \varphi_U = \pi$, i.e. $\varphi_U(E_q) \subset \{q\} \times \mathbb{R}^r$ for all $q \in U$:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi_U} & U \times \mathbb{R}^r \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\mathrm{pr}_1} & \\ & U & \end{array}$$

Examples

Example 1 (Product bundle)**.** Let V be a real vector space of dimension r and

$$\pi = \mathrm{pr}_1\colon E = M \times V \to M.$$

(i) $E_p = \{p\} \times V$, a vector space via V.

(ii) Choose a linear isomorphism $L\colon V \to \mathbb{R}^r$. Then one *global* trivialisation ($U = M$) is

$$\mathrm{id}_M \times L\colon M \times V \to M \times \mathbb{R}^r, \qquad (p, v) \mapsto (p, Lv).$$

E.g. the *cylinder* $S^1 \times \mathbb{R} \to S^1$.

Example 2 (Tangent bundle)**.** Let M be a manifold of dimension n and

$$TM := \{(p, v) \mid p \in M,\ v \in T_pM\}, \qquad \pi(p, v) = p.$$

(i) $(TM)_p = \{p\} \times T_pM \cong T_pM$, of dimension n; so the rank is n.

(ii) A chart $(U, x^1, \dots, x^n)$ gives the trivialisation

$$\varphi_U\Big(p,\ \sum_{i=1}^{n} v^i \,\frac{\partial}{\partial x^i}\Big|_p\Big) = (p, v^1, \dots, v^n) \in U \times \mathbb{R}^n.$$

It is linear on each fibre: it takes coordinates with respect to the basis $\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p$.

Example 3 (Cotangent bundle)**.**

$$T^*M := \{(p, \alpha) \mid p \in M,\ \alpha \in T_p^*M\}, \qquad \pi(p, \alpha) = p,$$

with fibres T_p^*M of dimension n and trivialisations from charts, using the dual basis $dx^1|_p, \dots, dx^n|_p$:

$$\varphi_U\Big(p,\ \sum_{i=1}^{n} a_i\, dx^i\big|_p\Big) = (p, a_1, \dots, a_n).$$

Example 4 (Möbius bundle)**.** A *line bundle*, i.e. a vector bundle of rank 1:

$$E := [0,1] \times \mathbb{R} \,\big/\, (0, t) \sim (1, -t), \qquad \pi\colon E \to S^1 := [0,1]/(0 \sim 1), \quad \pi([x,t]) = [x].$$

(i) Every fibre $E_{[x]} = \{[x, t] \mid t \in \mathbb{R}\}$ is a line. At the glued point, $E_{[0]} = \{[0, t]\} = \{[1, -t]\}$, and the vector space structure is well defined because $t \mapsto -t$ is linear.

(ii) Two trivialisations, over $U_1 = S^1 \setminus \{[0]\}$ and $U_2 = S^1 \setminus \{[\frac{1}{2}]\}$:

$$\varphi_1([x,t]) = ([x],\, t), \qquad \varphi_2([x,t]) = \begin{cases} ([x],\ \ t), & x \in (\frac{1}{2}, 1], \\ ([x], -t), & x \in [0, \frac{1}{2}). \end{cases}$$

φ_2 is well defined at the glued point: $[0, t] \mapsto ([0], -t)$ and $[1, -t] \mapsto ([1], -t) = ([0], -t)$. Both are linear on each fibre ($t \mapsto \pm t$), and U_1, U_2 cover S^1.

Bundle maps and trivial bundles

Definition 7.5 (Bundle map)**.** Let $\pi_E\colon E \to M$ and $\pi_F\colon F \to N$ be C^∞ vector bundles. A C^∞ *bundle map* from E to F is a pair of C^∞ maps $(\phi\colon E \to F,\ \underline{\phi}\colon M \to N)$ such that

(i) the diagram commutes, i.e. $\pi_F \circ \phi = \underline{\phi} \circ \pi_E$:

$$\begin{array}{ccc} E & \xrightarrow{\ \phi\ } & F \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi_F} \\ M & \xrightarrow[\ \underline{\phi}\ ]{} & N \end{array}$$

(ii) ϕ restricts to a linear map $\phi_p\colon E_p \to F_{\underline{\phi}(p)}$ for each $p \in M$.

Example (differential). A smooth map $f\colon M \to N$ induces the bundle map

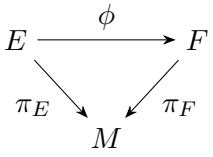
$$df\colon TM \to TN, \qquad df(p, v) = \big(f(p),\, df_p(v)\big), \qquad \text{over } \underline{\phi} = f.$$

(i) $df(p,v)$ lies over $f(p)$, so $\pi_{TN} \circ df = f \circ \pi_{TM}$.

(ii) $df_p \colon T_pM \to T_{f(p)}N$ is linear.

Special case: bundle maps over M. If E and F are bundles over the same M and $\underline{\phi} = \mathrm{id}_M$, then ϕ is a *bundle map over M*: conditions (i), (ii) become

(i′) $\pi_F \circ \phi = \pi_E$, i.e. $\phi(E_p) \subset F_p$:



(ii′) $\phi_p \colon E_p \to F_p$ is linear for every $p \in M$.

ϕ is a *bundle isomorphism over M* if there is a bundle map $\psi \colon F \to E$ over M with $\psi \circ \phi = \mathrm{id}_E$ and $\phi \circ \psi = \mathrm{id}_F$. Then E and F are *isomorphic over M*, $E \cong F$.
Example. Every trivialisation $\varphi_U \colon E|_U \to U \times \mathbb{R}^r$ is a bundle isomorphism over U:

(i′) φ_U is fibre-preserving, i.e. $\mathrm{pr}_1 \circ \varphi_U = \pi$ (Def. 7.1(ii)).

(ii′) φ_U is linear on each fibre (Def. 7.1(ii)).

Its inverse φ_U^{-1} is smooth (as φ_U is a diffeomorphism), fibre-preserving and linear on fibres, so it is again a bundle map.

Definition 7.6 (Trivial bundle)**.** A vector bundle $\pi \colon E \to M$ is *trivial* if it is isomorphic over M to the product bundle $M \times \mathbb{R}^r \to M$.

Back to the examples

- *Product bundle:* trivial, since $\mathrm{id}_M \times L$ is an isomorphism. In particular the cylinder $S^1 \times \mathbb{R}$ is trivial.
- *Möbius bundle:* locally trivial by φ_1, φ_2. Is it trivial? *No*, as we will prove with frames.

Sections

Definition (Section)**.** Let $\pi \colon E \to M$ be a vector bundle and $U \subset M$ open. A *section of E over U* is a map $s \colon U \to E$ with

$$\pi \circ s = \mathrm{id}_U, \qquad \text{i.e.} \quad s(p) \in E_p \ \text{ for all } p \in U.$$

We write

$$\Gamma(U,E) := \{s \colon U \to E \mid s \text{ is a } C^\infty \text{ section of } E \text{ over } U\}, \qquad \Gamma(E) := \Gamma(M,E).$$

Module structure. For $s, t \in \Gamma(U,E)$ and $f \in C^\infty(U)$ define pointwise

$$(s+t)(p) := s(p) + t(p), \qquad (fs)(p) := f(p)\, s(p) \qquad \text{(computed in } E_p).$$

Since the operations are defined pointwise in the vector spaces E_p, this makes $\Gamma(U,E)$ a module over the ring $C^\infty(U)$ (in particular a real vector space).

Example 5 (Product line bundle)**.** A section of $M \times \mathbb{R} \to M$ has the form $s(p) = (p, f(p))$ with $f = \mathrm{pr}_2 \circ s$. This gives a one-to-one correspondence

$$\{\text{sections of } M \times \mathbb{R} \to M\} \longleftrightarrow \{\text{functions } f\colon M \to \mathbb{R}\}, \qquad s \longleftrightarrow f = \mathrm{pr}_2 \circ s,$$

so we may identify

$$\Gamma(M \times \mathbb{R}) \cong C^\infty(M), \qquad \text{and likewise} \qquad \Gamma(M \times \mathbb{R}^r) \cong C^\infty(M, \mathbb{R}^r).$$

Example 6 (Tangent and cotangent bundle)**.** A section of TM assigns to each p a tangent vector $X_p \in T_pM$, i.e. it is a vector field; a section of T^*M is a 1-form:

$$\Gamma(TM) = \mathfrak{X}(M), \qquad \Gamma(T^*M) = \Omega^1(M).$$

Example 7 (Möbius bundle)**.** Recall $E = [0,1] \times \mathbb{R}/(0,t) \sim (1,-t)$ and $\pi([x,t]) = [x] \in S^1 = [0,1]/(0 \sim 1)$: the point $[x]$ of the base is given by $x \in [0,1]$, and $[x,t]$ is the vector with coordinate t in the fibre $E_{[x]}$.

- *What a section is here:* s assigns to each $[x] \in S^1$ a vector $s([x]) \in E_{[x]}$, so $s([x]) = [x, f(x)]$ for a unique number $f(x) \in \mathbb{R}$. So s is determined by a function $f\colon [0,1] \to \mathbb{R}$.
- *Condition at the glued point:* $[0] = [1]$ is one point of S^1, so $s([0]) = s([1])$, i.e. $[0, f(0)] = [1, f(1)]$. Since $[0,t] = [1,-t]$, this means
$$f(1) = -f(0).$$
(Extending f to $\mathbb{R}$ by $f(x+1) = -f(x)$ gives an *antiperiodic* function.)
- *Conversely:* every smooth f with this property (antiperiodic) defines a section $s_f([x]) := [x, f(x)]$. So
$$\{\text{sections of the Möbius bundle}\} \longleftrightarrow \{f\colon \mathbb{R} \to \mathbb{R} \text{ smooth},\ f(x+1) = -f(x)\}.$$
- *Example:* $f(x) = \cos(\pi x)$, since $\cos(\pi x + \pi) = -\cos(\pi x)$. The section s_f vanishes exactly at $[\frac{1}{2}]$.
- *Every section has a zero:* $f(0)$ and $f(1) = -f(0)$ have opposite signs (or are 0), so by the intermediate value theorem $f(x_0) = 0$ for some $x_0 \in [0,1]$, i.e. $s([x_0]) = 0$.

Frames

Definition (Frame)**.** A *frame* for a vector bundle $E \to M$ of rank r over an open set $U \subset M$ is a collection of sections $e_1, \dots, e_r \in \Gamma(U, E)$ such that for every $p \in U$ the vectors $e_1(p), \dots, e_r(p)$ form a basis of the fibre E_p. If $U = M$, it is a *global frame.*

Examples.

- For a chart $(U, x^1, \dots, x^n)$: $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ is a frame of TM over U, and $dx^1, \dots, dx^n$ is a frame of T^*M over U.
- The constant sections $p \mapsto (p, v_i)$, $v_1, \dots, v_r$ the standard basis of $\mathbb{R}^r$, form a global frame of $M \times \mathbb{R}^r$.

Proposition 7.22. *A C^∞ vector bundle $\pi\colon E \to M$ is trivial if and only if it has a C^∞ global frame.*

Proof. ($\Rightarrow$) Let $\phi\colon E \to M \times \mathbb{R}^r$ be a bundle isomorphism over M (a trivialisation) and $v_1, \dots, v_r$ the standard basis of $\mathbb{R}^r$. Define

$$e_i(p) := \phi^{-1}(p, v_i), \qquad i = 1, \dots, r.$$

Then e_i is a C^∞ section *(as ϕ^{-1} is smooth and fibre-preserving, $e_i(p) \in E_p$)*, and since $\phi^{-1}\colon \{p\} \times \mathbb{R}^r \to E_p$ is a linear isomorphism, $e_1(p), \dots, e_r(p)$ is a basis of E_p. So $e_1, \dots, e_r$ is a global frame.

($\Leftarrow$) Let $e_1, \dots, e_r$ be a C^∞ global frame. Every $e \in E_p$ can be written uniquely as $e = \sum_i a^i(e)\, e_i(p)$. Define

$$\phi\colon E \to M \times \mathbb{R}^r, \qquad \phi(e) := \big(\pi(e),\, a^1(e), \dots, a^r(e)\big).$$

ϕ is fibre-preserving and, on each fibre, it takes coordinates with respect to the basis $e_1(p), \dots, e_r(p)$, so it is a linear isomorphism $E_p \to \{p\} \times \mathbb{R}^r$. Its inverse is

$$\phi^{-1}(p, a) = \sum_i a^i\, e_i(p),$$

which is C^∞ because the e_i are; ϕ is C^∞ because the coefficients a^i of a C^∞ frame are C^∞. Hence ϕ is a bundle isomorphism over M, and E is trivial. □

Corollary. *A line bundle ($r = 1$) is trivial if and only if it has a nowhere-vanishing C^∞ section.*

Proof. For $r = 1$, a frame is a single section e_1 with $e_1(p) \neq 0$ for all p. □

Example 8 (The Möbius bundle is not trivial)**.** By Example 7 every section of the Möbius bundle has a zero. So it has no nowhere-vanishing section and, by the corollary, it is not trivial. The cylinder $S^1 \times \mathbb{R}$ is trivial, with global frame $[x] \mapsto ([x], 1)$.

Transition functions

Definition (Transition functions)**.** Let $\{(U_\alpha, \varphi_\alpha)\}$ be a trivialising open cover of E. For α, β with $U_\alpha \cap U_\beta \neq \emptyset$, the map

$$\varphi_\alpha \circ \varphi_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r$$

is fibre-preserving and linear on each fibre. Hence there is a unique smooth map $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to GL(r, \mathbb{R})$ with

$$\varphi_\alpha \circ \varphi_\beta^{-1}(p, v) = \big(p,\ g_{\alpha\beta}(p)\, v\big) \qquad (p \in U_\alpha \cap U_\beta,\ v \in \mathbb{R}^r).$$

The maps $g_{\alpha\beta}$ are called the *transition functions* of E (with respect to the cover).

Meaning. Compare with the identity:

$$\varphi_\alpha \circ \varphi_\beta^{-1}(p, v) = \big(\ \underbrace{p}_{\text{unchanged}}\ ,\ \underbrace{g_{\alpha\beta}(p)\, v}_{\text{instead of } v}\ \big).$$

The point p stays the same; only the fibre coordinate v is corrected by the matrix $g_{\alpha\beta}(p)$. If the two trivialisations agree on $U_\alpha \cap U_\beta$, then $g_{\alpha\beta}(p)\, v = v$ for all v, i.e. $g_{\alpha\beta} = \mathbb{1}$.

Example 9 (Möbius bundle)**.** For the trivialisations φ_1, φ_2 of Example 4, the overlap

$$U_1 \cap U_2 = \{[x] \mid x \in (0, \tfrac{1}{2})\} \ \cup\ \{[x] \mid x \in (\tfrac{1}{2}, 1)\}$$

has two components. Compute $\varphi_1 \circ \varphi_2^{-1}([x], t)$:

- for $x \in (\frac{1}{2}, 1)$: $\varphi_2^{-1}([x], t) = [x, t]$, so $\varphi_1 \circ \varphi_2^{-1}([x], t) = ([x],\, t)$;
- for $x \in (0, \frac{1}{2})$: $\varphi_2^{-1}([x], t) = [x, -t]$, so $\varphi_1 \circ \varphi_2^{-1}([x], t) = ([x], -t)$.

Hence

$$g_{12}([x]) = \begin{cases} +1, & x \in (\frac{1}{2}, 1), \\ -1, & x \in (0, \frac{1}{2}), \end{cases} \qquad g_{12}\colon U_1 \cap U_2 \to GL(1, \mathbb{R}) = \mathbb{R} \setminus \{0\}.$$

Part II: Bundle metrics, connections, and parallel transport

Definition

Definition (Riemannian metric on a vector bundle)**.** A *Riemannian metric* (or *bundle metric*) on a vector bundle $\pi\colon E \to M$ assigns to every $p \in M$ an inner product

$$\langle\ ,\ \rangle_p\colon E_p \times E_p \to \mathbb{R}$$

on the fibre E_p (bilinear, symmetric, positive definite), such that for all $s, t \in \Gamma(E)$ the function

$$\langle s, t\rangle\colon M \to \mathbb{R}, \qquad p \mapsto \langle s(p), t(p)\rangle_p$$

is C^∞. A vector bundle together with a Riemannian metric is a *Riemannian bundle*.

A Riemannian metric on M is exactly a Riemannian metric on the tangent bundle TM.

The trivial case

Example 10.7 (Induced metric on a trivial bundle)**.** Let $\phi\colon E \to M \times \mathbb{R}^r$ be a trivialisation and $\phi_p\colon E_p \to \mathbb{R}^r$ the linear isomorphism on the fibre over p. Define

$$\langle u, v\rangle_p := \big\langle \phi_p(u), \phi_p(v)\big\rangle_{\mathbb{R}^r} \qquad (u, v \in E_p),$$

with the Euclidean inner product on $\mathbb{R}^r$.

- *Inner product on E_p:* bilinear and symmetric since ϕ_p is linear; positive definite since ϕ_p is injective, so $u \neq 0 \Rightarrow \phi_p(u) \neq 0$.
- *Smooth:* for $s, t \in \Gamma(E)$ write $h = \mathrm{pr}_2 \circ \phi \circ s$ and $k = \mathrm{pr}_2 \circ \phi \circ t$, which are smooth maps $M \to \mathbb{R}^r$. Then
$$\langle s, t\rangle(p) = \langle s(p), t(p)\rangle_p = \langle h(p), k(p)\rangle_{\mathbb{R}^r} = \sum_{i=1}^{r} h^i(p)\, k^i(p),$$
which is smooth in p.

Existence of a bundle metric

Theorem 10.8. *On every C^∞ vector bundle $\pi\colon E \to M$ there is a Riemannian metric.*

Proof. Let $\{U_\alpha\}$ be a trivialising open cover for E. On $E|_{U_\alpha}$ there is the metric $\langle\ ,\ \rangle_\alpha$ induced from the trivial bundle $U_\alpha \times \mathbb{R}^r$ (Example 10.7). Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Then

$$\boxed{\langle\ ,\ \rangle := \sum_\alpha \rho_\alpha \langle\ ,\ \rangle_\alpha}$$

is a finite sum in a neighbourhood of each point (local finiteness) and is a Riemannian metric on E. □

Connections

Definition 10.1 (Connection)**.** A *connection* on a vector bundle $E \to M$ is a map

$$\nabla\colon \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \qquad (X, s) \mapsto \nabla_X s,$$

such that for all $X, Y \in \mathfrak{X}(M)$, $s, t \in \Gamma(E)$, $f, g \in C^\infty(M)$, $a, b \in \mathbb{R}$:

(i) $\nabla_{fX+gY}\, s = f\, \nabla_X s + g\, \nabla_Y s$ ($C^\infty(M)$-linear in X)

(ii) $\nabla_X(as + bt) = a\, \nabla_X s + b\, \nabla_X t$ ($\mathbb{R}$-linear in s)

(iii) $\nabla_X(fs) = (Xf)\, s + f\, \nabla_X s$ (Leibniz rule)

The connection induced by a trivialisation

Example 10.3 (Induced connection)**.** Let $\phi\colon E \to M \times \mathbb{R}^r$ be a trivialisation, $v_1, \dots, v_r$ the standard basis of $\mathbb{R}^r$, and

$$e_i(p) := \phi^{-1}(p, v_i) \qquad (p \in M,\ i = 1, \dots, r)$$

the corresponding global frame (Prop. 7.22). Every $s \in \Gamma(E)$ is $s = \sum_i h^i e_i$, where the coefficient vector is

$$h = (h^1, \dots, h^r) := \mathrm{pr}_2 \circ \phi \circ s\colon M \to \mathbb{R}^r,$$

since $\phi(s(p)) = \big(p, \sum_i h^i(p)\, v_i\big)$ (ϕ is linear on fibres and $\phi(e_i(p)) = (p, v_i)$). In particular the h^i are C^∞.

$$\boxed{\nabla_X\Big(\sum_i h^i e_i\Big) := \sum_i (Xh^i)\, e_i}$$

This is a connection: (i) and (ii) are clear, so we only check the Leibniz rule. For $f \in C^\infty(M)$, $fs = \sum_i (fh^i) e_i$, and by the product rule

$$\nabla_X(fs) = \sum_i X(fh^i)\, e_i = \sum_i \big((Xf)h^i + f\, Xh^i\big) e_i = (Xf)\, s + f\, \nabla_X s.$$

Existence of a connection

For connections ∇^0, ∇^1 on E, the Leibniz rule gives

$$(\nabla^0_X + \nabla^1_X)(fs) = \mathbf{2}\,(Xf)\, s + f\,(\nabla^0_X + \nabla^1_X)s,$$

so $\nabla^0 + \nabla^1$ is *not* a connection. But the factor becomes 1 if the coefficients add up to 1:

Proposition 10.5. *If $\nabla^1, \dots, \nabla^m$ are connections on E and $t_1, \dots, t_m \in \mathbb{R}$ with $\sum_i t_i = 1$, then $\sum_i t_i \nabla^i$ is a connection.*

Proof. Let $\nabla := \sum_i t_i \nabla^i$, i.e. $\nabla_X s := \sum_i t_i\, \nabla^i_X s$.
(i), (ii): each ∇^i is $C^\infty(M)$-linear in X and $\mathbb{R}$-linear in s, and a linear combination of such maps with constant coefficients t_i is again.
(iii) Leibniz rule: for $f \in C^\infty(M)$,

$$\begin{aligned}
\nabla_X(fs) &= \sum_i t_i\, \nabla^i_X(fs) && \text{(definition of } \nabla) \\
&= \sum_i t_i \big((Xf)\, s + f\, \nabla^i_X s\big) && \text{(Leibniz rule for each } \nabla^i) \\
&= \Big(\sum_i t_i\Big)(Xf)\, s + f \sum_i t_i\, \nabla^i_X s && \text{(sort the terms)} \\
&= (Xf)\, s + f\, \nabla_X s && \Big(\textstyle\sum_i t_i = 1,\ \text{definition of } \nabla\Big).
\end{aligned}$$

□

Theorem 10.6. *Every C^∞ vector bundle $E \to M$ has a connection.*

Proof. Fix a trivialising open cover $\{U_\alpha\}$ for E and a partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$ *(ρ_α smooth, $\rho_\alpha \geq 0$, $\operatorname{supp}\rho_\alpha \subset U_\alpha$, the supports are locally finite, $\sum_\alpha \rho_\alpha = 1$).* On each U_α the bundle $E|_{U_\alpha}$ is trivial and so has a connection ∇^α by Example 10.3. For $X \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$ denote by s_α the restriction of s to U_α and define

$$\boxed{\nabla_X s := \sum_\alpha \rho_\alpha \, \nabla^\alpha_X s_\alpha}$$

(each term extended by 0 outside U_α; this is smooth since $\operatorname{supp}\rho_\alpha \subset U_\alpha$).

Although this looks like an infinite sum, it is a finite sum in a neighbourhood of each point p: by local finiteness there is an open neighbourhood U of p that meets only finitely many $\operatorname{supp}\rho_\alpha$, so on U all but finitely many ρ_α are zero, and $\nabla_X s$ is a finite sum on U with $\sum_\alpha \rho_\alpha = 1$.

By checking on U: $\nabla_X s$ is C^∞-linear in X and $\mathbb{R}$-linear in s, since each term is, and it satisfies the Leibniz rule:

$$\nabla_X(fs) = \sum_\alpha \rho_\alpha\big((Xf)\,s + f\,\nabla^\alpha_X s\big) = \Big(\sum_\alpha \rho_\alpha\Big)(Xf)\,s + f\,\nabla_X s = (Xf)\,s + f\,\nabla_X s.$$

Hence ∇ is a connection on E. □

The connection matrix

Let $e_1, \ldots, e_r$ be a frame of E over U. Every $s \in \Gamma(U, E)$ is a linear combination $s = \sum_j h^j e_j$, so $\nabla_X s$ can be computed from the $\nabla_X e_j$ by linearity and the Leibniz rule.

Definition (Connection matrix)**.** Let $e = \begin{pmatrix} e_1 & \cdots & e_r \end{pmatrix}$ be a frame of E over U. Since $\nabla_X e_j \in \Gamma(U, E)$ is a section and $e_1(p), \ldots, e_r(p)$ is a basis of every fibre E_p, we can write $\nabla_X e_j$ uniquely in the frame:

$$\boxed{\nabla_X e_j = \sum_{i=1}^r \omega^i_j(X)\, e_i} \qquad \text{i.e.} \qquad \nabla e = e\,\omega.$$

The ω^i_j are the *connection forms* and $\omega = [\omega^i_j]$ is the *connection matrix* of ∇ relative to e.

Each ω^i_j is a 1-form on U, because $\nabla_X e_j$ is C^∞-linear in X.

Local formula. For $s = \sum_j h^j e_j = e\,h$, the Leibniz rule gives

$$\nabla_X s = \sum_j \Big((Xh^j)\,e_j + h^j\,\nabla_X e_j\Big) = \sum_i \Big(Xh^i + \sum_j \omega^i_j(X)\,h^j\Big) e_i,$$

$$\text{i.e.} \qquad \boxed{\nabla(e\,h) = e\,\big(dh + \omega\,h\big)} \qquad \text{“locally } \nabla = d + \omega\text{”}.$$

Conversely. Every $r \times r$ matrix ω of 1-forms on U defines by this formula a connection on $E|_U$ with connection matrix ω. So on a framed open set

$$\{\text{connections on } E|_U\} \;\longleftrightarrow\; \{r \times r \text{ matrices of 1-forms on } U\}.$$

Examples.

- Induced connection (Example 10.3), in its own frame: $\nabla e = 0$, so $\omega = 0$.
- Line bundles ($r = 1$): the frame is one section e, ω is a single 1-form with $\nabla_X e = \omega(X)\,e$, and $s = h\,e$ with $h \in C^\infty(U)$. Then

$$\nabla_X(h\,e) = (Xh)\,e + h\,\nabla_X e = (Xh)\,e + h\,\omega(X)\,e = \big(Xh + \omega(X)\,h\big)\,e,$$

  i.e. $\nabla(h\,e) = e\,(dh + \omega\,h)$.

Example (Flat $\mathbb{R}^2$ in polar coordinates)**.** On $T\mathbb{R}^2$ let D be the connection induced by the standard frame (∂_x, ∂_y), i.e. $D_X(a\,\partial_x + b\,\partial_y) = (Xa)\,\partial_x + (Xb)\,\partial_y$; relative to (∂_x, ∂_y) its matrix is 0. On $U = \mathbb{R}^2 \setminus \{0\}$ take the polar frame

$$e_r = \cos\theta\,\partial_x + \sin\theta\,\partial_y, \qquad e_\theta = -\sin\theta\,\partial_x + \cos\theta\,\partial_y.$$

Differentiating the coefficients, with $X(\cos\theta) = -\sin\theta\,d\theta(X)$ and $X(\sin\theta) = \cos\theta\,d\theta(X)$:

$$\begin{aligned} D_X e_r &= d\theta(X)\big(-\sin\theta\,\partial_x + \cos\theta\,\partial_y\big) = d\theta(X)\,e_\theta, \\ D_X e_\theta &= d\theta(X)\big(-\cos\theta\,\partial_x - \sin\theta\,\partial_y\big) = -d\theta(X)\,e_r. \end{aligned}$$

In general, by definition of the connection matrix (column j = coefficients of $D_X e_j$),

$$\begin{aligned} D_X e_r &= \omega_r^r(X)\,e_r + \omega_r^\theta(X)\,e_\theta, \\ D_X e_\theta &= \omega_\theta^r(X)\,e_r + \omega_\theta^\theta(X)\,e_\theta. \end{aligned}$$

Comparing coefficients: $\omega_r^r = 0$, $\omega_r^\theta = d\theta$, $\omega_\theta^r = -d\theta$, $\omega_\theta^\theta = 0$. So

$$\omega = \begin{pmatrix} \omega_r^r & \omega_\theta^r \\ \omega_r^\theta & \omega_\theta^\theta \end{pmatrix} = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix} \neq 0.$$

Change of frame: the transformation law

Setup. Let $e = \begin{pmatrix} e_1 & \cdots & e_r \end{pmatrix}$ and $\bar{e} = \begin{pmatrix} \bar{e}_1 & \cdots & \bar{e}_r \end{pmatrix}$ be two frames of E over U, with connection matrices ω and $\bar{\omega}$:

$$\nabla e = e\,\omega, \qquad \nabla \bar{e} = \bar{e}\,\bar{\omega}.$$

Each $\bar{e}_\ell$ is a linear combination of the e_k:

$$\bar{e}_\ell = \sum_{k=1}^{r} e_k\, a_\ell^k, \qquad \text{i.e.} \qquad \bar{e} = e\,a, \qquad a = [a_\ell^k] \colon U \to GL(r, \mathbb{R}).$$

- a is smooth: the a_ℓ^k are the coefficients of the smooth sections $\bar{e}_\ell$ in the smooth frame e.
- $a(p)$ is invertible: it is the change-of-basis matrix between the bases $e(p)$ and $\bar{e}(p)$ of E_p. Hence $e = \bar{e}\,a^{-1}$.

We write $da := [da_\ell^k]$ for the matrix of 1-forms.

Leibniz rule in matrix form. With the function written on the right, the Leibniz rule reads $\nabla(s\,f) = (\nabla s)\,f + s\,df$. Applied to each entry $\bar{e}_\ell = \sum_k e_k\,a_\ell^k$, this gives

$$\nabla(e\,a) = (\nabla e)\,a + e\,da.$$

Theorem (Transformation law)**.** *If $\bar{e} = e\,a$ as above, then*

$$\boxed{\bar{\omega} = a^{-1}\omega\,a + a^{-1}da}$$

Proof.

$$\begin{aligned} \nabla\bar{e} &= \nabla(e\,a) \\ &= (\nabla e)\,a + e\,da && \text{(Leibniz rule)} \\ &= e\,\omega\,a + e\,da && (\nabla e = e\,\omega) \\ &= \bar{e}\,a^{-1}\omega\,a + \bar{e}\,a^{-1}da && (e = \bar{e}\,a^{-1}) \\ &= \bar{e}\,\big(a^{-1}\omega\,a + a^{-1}da\big). \end{aligned}$$

On the other hand $\nabla\bar{e} = \bar{e}\,\bar{\omega}$. Since the coefficients with respect to the frame $\bar{e}$ are unique, $\bar{\omega} = a^{-1}\omega\,a + a^{-1}da$. □

Example.

- *Polar frame.* Connection D on $T\mathbb{R}^2$ restricted to $U = \mathbb{R}^2 \setminus \{0\}$.
  - *Start:* the standard frame $e = (\partial_x, \partial_y)$. Here $\omega = 0$: $T\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}^2$ is a trivial bundle, and D is the connection induced by the trivialisation given by exactly this frame (Example 10.3), so $D\partial_x = D\partial_y = 0$.
  - *End:* the polar frame $\bar{e} = (e_r, e_\theta) = (\partial_x, \partial_y)\, R(\theta)$, i.e. $\bar{e} = e\, a$ with

$$a = R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

  By the transformation law with $\omega = 0$:

$$\bar{\omega} = a^{-1} \cdot 0 \cdot a + a^{-1} da = R(\theta)^{-1} dR(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix} d\theta = \begin{pmatrix} 0 & -d\theta \\ d\theta & 0 \end{pmatrix},$$

  the matrix computed directly before.

Outlook: the curvature matrix (no proof). Relative to a frame e, the *curvature matrix* of ∇ is the matrix of 2-forms

$$\Omega := d\omega + \omega \wedge \omega, \qquad \text{i.e.} \qquad \Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j.$$

Under a change of frame $\bar{e} = e\, a$ it transforms by conjugation only:

$$\boxed{\bar{\Omega} = a^{-1}\, \Omega\, a}.$$

Parallel transport

Let ∇ be a connection on $E \to M$ and $c\colon [a,b] \to M$ a smooth curve.

Definition (Section along a curve)**.** A *section of E along c* is a map $s\colon [a,b] \to E$ with $s(t) \in E_{c(t)}$ for all t. It is smooth if in every local frame e its coefficients $s^i(t)$ in $s(t) = \sum_i s^i(t)\, e_i(c(t))$ are smooth. We write $\Gamma(c^*E)$ for the smooth sections along c.

Theorem (Covariant derivative along c)**.** *There is a unique $\mathbb{R}$-linear map $\dfrac{D}{dt}\colon \Gamma(c^*E) \to \Gamma(c^*E)$ with*

(i) $\dfrac{D(fs)}{dt} = \dfrac{df}{dt}\, s + f\, \dfrac{Ds}{dt}$ *for $f \in C^\infty([a,b])$;*

(ii) $\dfrac{Ds}{dt}(t) = \nabla_{c'(t)} \tilde{s}$ *if $s = \tilde{s} \circ c$ with $\tilde{s} \in \Gamma(E)$.*

In a frame e with connection matrix ω:

$$\boxed{\frac{Ds}{dt} = \sum_i \Big(\dot{s}^i + \sum_j \omega^i_j\big(c'(t)\big)\, s^j \Big) e_i} \qquad \textit{i.e.} \qquad \frac{D(e\,h)}{dt} = e\,\big(\dot{h} + \omega(c')\, h\big).$$

Here we used, as for the local formula, the Leibniz rule (i) and the connection matrix:

$$\frac{D}{dt}\big(s^j e_j\big) = \dot{s}^j\, e_j + s^j\, \nabla_{c'} e_j, \qquad \nabla_{c'} e_j = \sum_i \omega^i_j(c')\, e_i.$$

Definition (Parallel section)**.** $s \in \Gamma(c^*E)$ is *parallel along* c if $\dfrac{Ds}{dt} \equiv 0$. In a frame this is the linear ODE

$$\boxed{\dot{s}^i(t) + \sum_j \omega^i_j\big(c'(t)\big)\, s^j(t) = 0 \quad (i = 1, \dots, r), \qquad \text{i.e.} \qquad \dot{h} + \omega(c'(t))\, h = 0.}$$

Theorem. *For every $v \in E_{c(a)}$ there is a unique parallel section s along c with $s(a) = v$, and the* parallel transport

$$P_c \colon E_{c(a)} \to E_{c(b)}, \qquad P_c(v) := s(b),$$

is a linear isomorphism.

Example (Trivial line bundle over $\mathbb{R}^2$)**. Setup.** Let $E = \mathbb{R}^2 \times \mathbb{R}$ with the global frame $e = 1$, and let ∇ be the connection with connection matrix (a single 1-form)

$$\omega = x\, dy, \qquad \text{i.e.} \qquad \nabla_X h = Xh + \omega(X)\, h \quad (h \in C^\infty(\mathbb{R}^2)).$$

The equation. For a curve c, a section along c is a function $h(t)$, and it is parallel iff

$$\dot{h}(t) = -\,\omega\big(c'(t)\big)\, h(t),$$

a linear scalar ODE. Its solution with $h(a) = v$ is

$$h(t) = v\, \exp\Big(-\int_a^t \omega\big(c'(\tau)\big)\, d\tau\Big), \qquad \text{so} \qquad P_c(v) = \exp\Big(-\int_c \omega\Big)\, v.$$

Two paths from $(0,0)$ **to** $(1,1)$**.** On a straight piece $c(t) = (x(t), y(t))$ we have $\omega(c'(t)) = x(t)\, \dot{y}(t)$.

- c_1: first right, then up.
  - $(t, 0)$, $t \in [0,1]$: $x\, \dot{y} = t \cdot 0 = 0$.
  - $(1, t)$, $t \in [0,1]$: $x\, \dot{y} = 1 \cdot 1 = 1$.

  So $\int_{c_1} \omega = 0 + 1 = 1$ and $\boxed{P_{c_1}(v) = e^{-1} v}$.
- c_2: first up, then right.
  - $(0, t)$, $t \in [0,1]$: $x\, \dot{y} = 0 \cdot 1 = 0$.
  - $(t, 1)$, $t \in [0,1]$: $x\, \dot{y} = t \cdot 0 = 0$.

  So $\int_{c_2} \omega = 0$ and $\boxed{P_{c_2}(v) = v}$.

Conclusion. $P_{c_1} \neq P_{c_2}$: parallel transport depends on the path, not only on the endpoints.